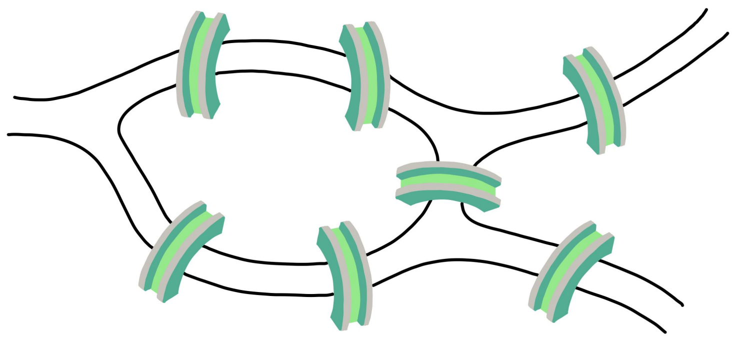
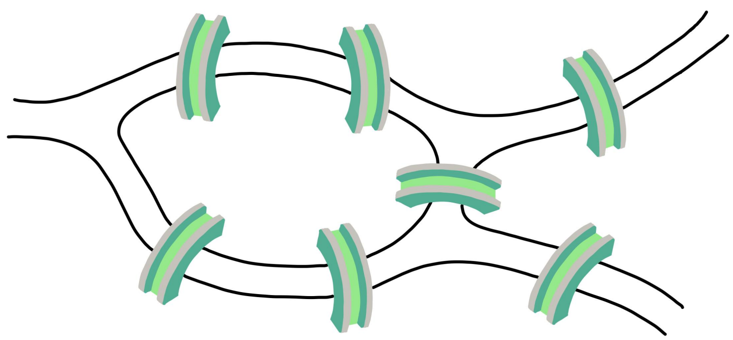
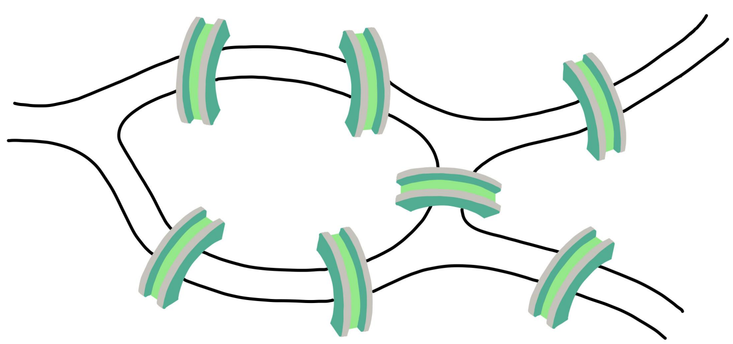
Name: ________________________

**Math Adventures**
**Week 2: Graph Theory**

1. **The Seven Bridges of Königsberg:** Can you draw a path across all seven bridges without crossing a bridge twice?



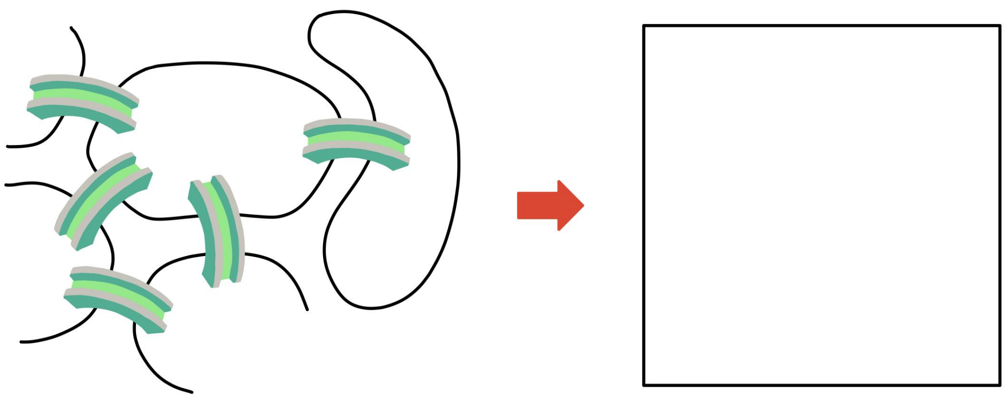




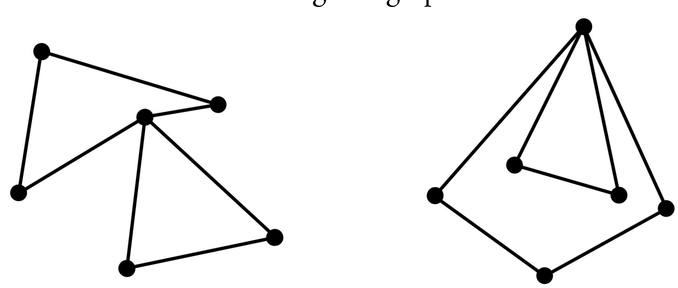
Leonhard Euler invented **graph theory** to solve the problem of the Seven Bridges. A **graph** is made up of points connected by lines. Here are the characteristics of a graph:

- Each point is called a **vertex**.
- Each line is called an **edge**.
- A single graph can be drawn in different ways as long as the vertices are connected in the same way.
- The edges can be any length and can go in any direction.

2. The following place is called "The Five Bridges of Doodletown." Redraw this place as a graph in the box provided.

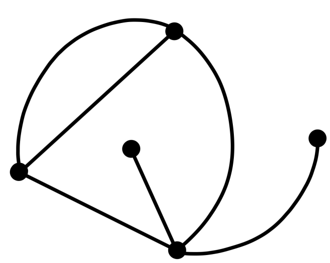


3. Is there a difference between the following two graphs?



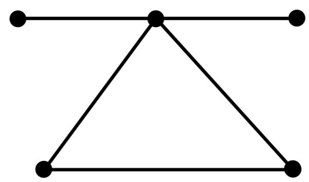
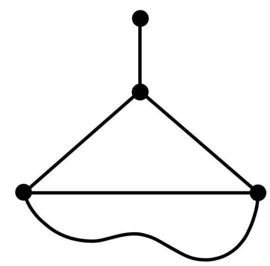
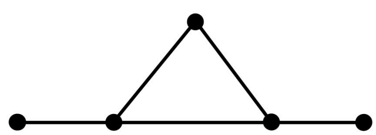
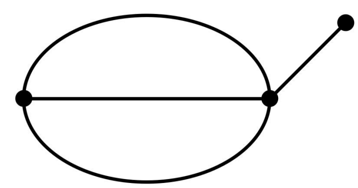
The **degree** of a vertex is the number of edges connecting to the vertex.

4. Label each vertex of the graph with its degree.



An **Euler path** is a path that uses every edge in a graph without retracing an edge.

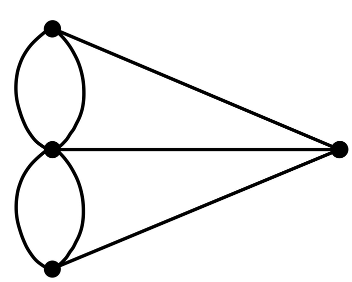
5. Two of the graphs below have Euler paths and two do not. Circle the ones that have Euler paths.

6. Now label the degrees of all the vertices in the graphs above.

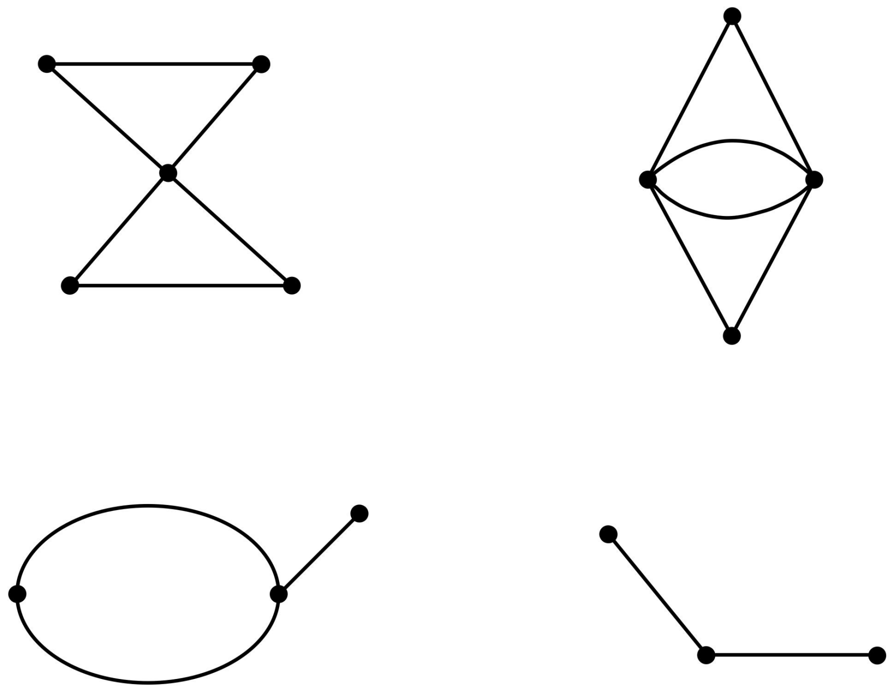
   If a graph has an Euler path, then one of two things can happen: (1) Every vertex of the graph has an even degree, or (2) every vertex has an even degree except for two vertices which have an odd degree. Does this rule match what you found above?

7. **Euler's solution:** Label each vertex of the graph of the seven bridges of Königsberg with its degree. Does the graph have an Euler path?



An **Euler circuit** is almost the same as an Euler path, but it must start and end at the same vertex.

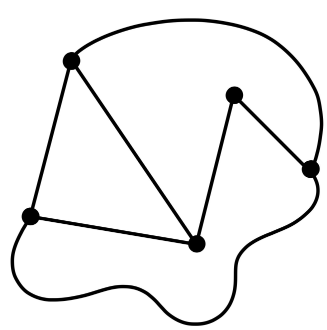
8. Two of the graphs below have Euler circuits and two do not. Circle the ones that have Euler circuits.



9. Now label the degrees of all the vertices in the graphs above.

   If a graph has an Euler circuit, what do you notice about the degree of each vertex in the graph?

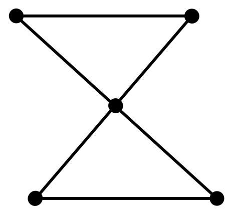
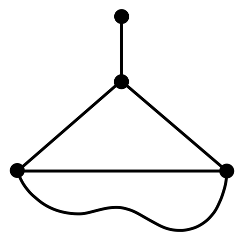
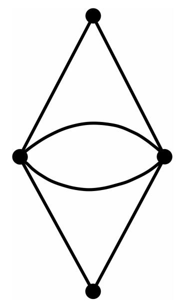
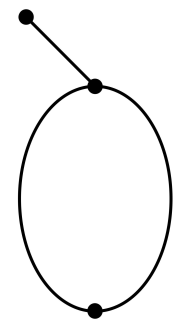
10. Label each vertex of the graph with its degree. Does this graph have an Euler circuit?



A **planar graph** is a graph that can be drawn without any edges crossing each other.

- A **face** of a planar graph is a region surrounded by edges. The region outside of the graph is also a face.

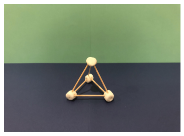
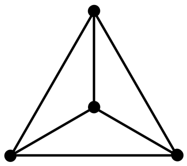
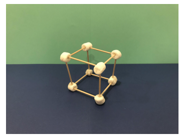
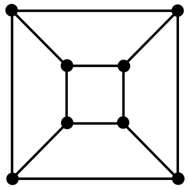
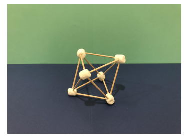
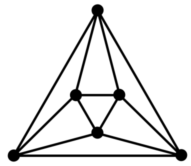
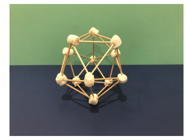
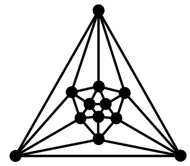
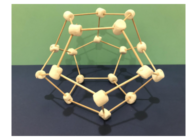
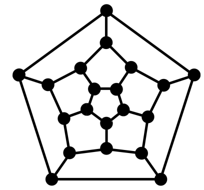
11. Fill out the table below for the following graphs. Remember to count the outside of each graph as a face!

| |  |  |  |  |
|---|---|---|---|---|
| Faces ($F$) | | | | |
| Vertices ($V$) | | | | |
| Edges ($E$) | | | | |
| $F + V - E =$ | | | | |

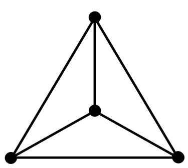
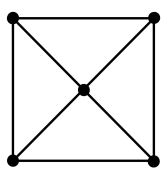
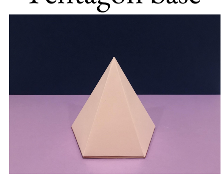
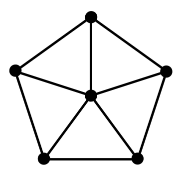
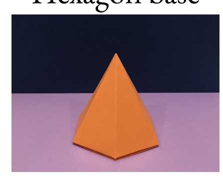
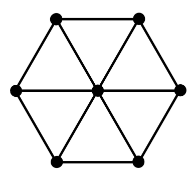
12. Looking at the last row of the table, we can develop a formula that is true for all <u>planar</u> graphs. It is called Euler's formula.

**Euler's formula:** $F + V - E =$

Euler’s formula is true for both graphs and polyhedra because every polyhedron can be “flattened” and drawn as a planar graph. Here are the graphs of the five Platonic solids we built last week:

| | Tetrahedron   | Cube   | Octahedron   | Icosahedron   | Dodecahedron   |
|---|---|---|---|---|---|
| Faces ($F$) | | | | | |
| Vertices ($V$) | | | | | |
| Edges ($E$) | | | | | |
| $F + V - E =$ | | | | | |

Here are the graphs of the four pyramids we looked at last week:

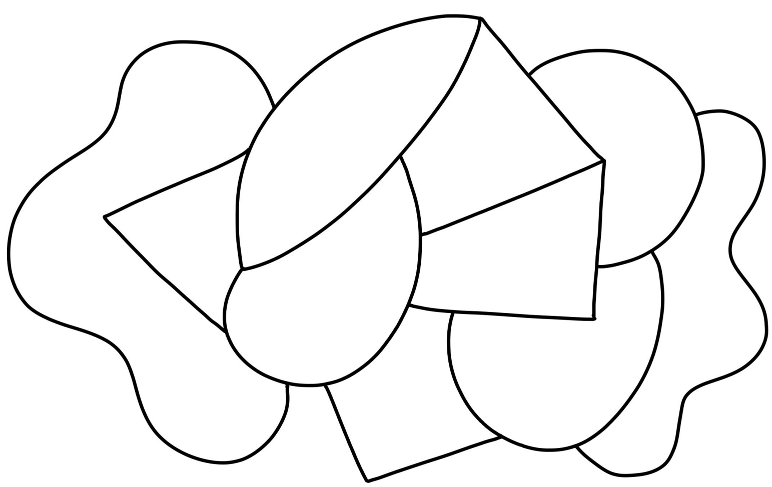
| | Triangle base   | Quadrilateral base   | Pentagon base   | Hexagon base   |
|---|---|---|---|---|
| Faces ($F$) | | | | |
| Vertices ($V$) | | | | |
| Edges ($E$) | | | | |
| $F + V - E =$ | | | | |

Graph theory is also used in coloring maps!

13. You are a map publisher and want to use as few colors as possible to print your maps in order to save money. Color in the map below so that countries that are touching have different colors. Try to use as few colors as possible. (Hint: Start from one country and color in adjacent countries as you go.)


How many colors did you have to use?

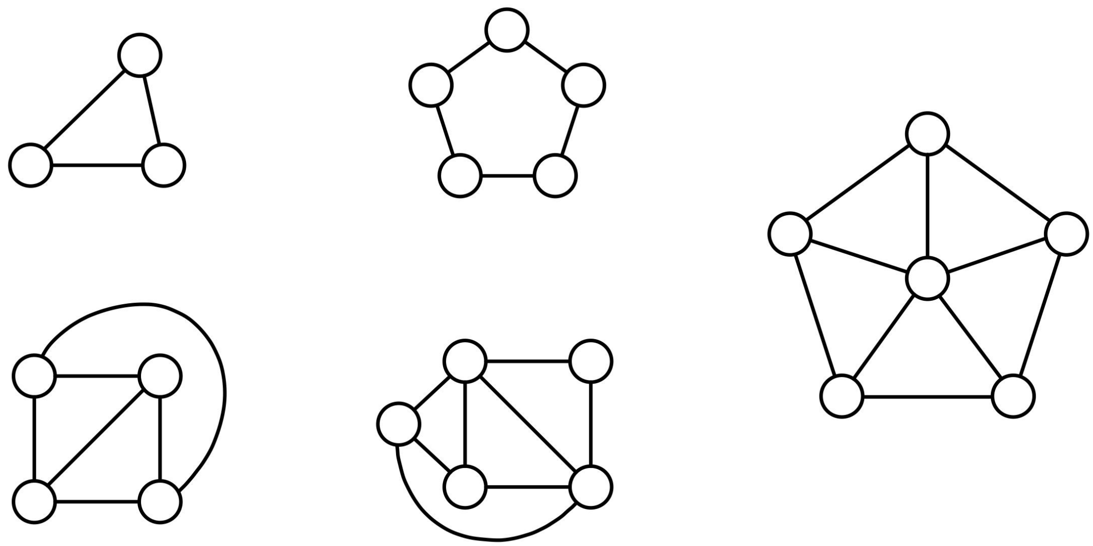
14. Color in the map below so that countries that are touching have different colors. Try to use as few colors as possible.



How many colors did you have to use?

The **chromatic number** of a graph is the smallest number of colors needed to color in the vertices so that connected vertices have different colors.

15. Color each vertex in the planar graphs below so that connected vertices have different colors. Try to use as few colors as possible. Next to each graph, write its chromatic number.



16. The chromatic number of a planar graph is at most ____.

**The Game of Sprouts:**

- Start with any number of dots.
- Players take turns.
- On a turn, a player connects two dots (or one dot with itself) and puts a dot in the middle of the line.
- Each dot can have at most three lines connecting to it.
- Lines are not allowed to cross.
- Eventually, a player will not be able to move, so the last player who is able to move wins!

Find a partner and play a game of Sprouts!

## Lesson Summary

Leonhard Euler developed **graph theory** to solve the problem of the Seven Bridges of Königsberg.

In graph theory, a **graph** is made up of vertices (points) connected by edges (lines). The edges can be any length and can go in any direction.

The **degree** of a vertex is the number of edges connecting to the vertex.

An **Euler path** is a path that uses every edge in a graph without retracing an edge.

- If a graph has an Euler path, each vertex of the graph has an even degree, except for two vertices which both have either an odd degree or an even degree.

An **Euler circuit** is a path that uses every edge in a graph without retracing an edge, starting and ending at the same vertex.

- If a graph has an Euler circuit, each vertex of the graph has an even degree.

A **planar** graph is a graph that can be drawn without any edges crossing each other.

- A **face** of a planar graph is a region surrounded by edges. The region outside of the graph is also a face.

For every planar graph, $F + V - E = 2$, where $F$ represents the number of faces, $V$ represents the number of vertices, and $E$ represents the number of edges. This formula is called **Euler's formula**.

- Euler's formula is true for both graphs and polyhedra because every polyhedron can be "flattened" and drawn as a planar graph.

The **chromatic number** of a graph is the smallest number of colors needed to color in the vertices so that connected vertices have different colors.

- The chromatic number of a planar graph is at most 4.

References: Olga Radko Endowed Math Circle archive, Windward School Math Department, Joel David Hamkins's website, *Discrete Mathematics and Its Applications* by Kenneth H. Rosen